CALCULUS III
FALL 2019
PROBLEM SET 5 EXERCISE 1 SOLUTION

1. $f(x, y) = y^2 - x^2$

For a,b,c see the drawings on the next page.

(a) The level sets contain the points (x, y) that satisfy $y^2 - x^2 = c$. When $c = 0$ we have $y^2 = x^2$, so we have $y = \pm x$. When $c > 0$ we have vertical hyperbolas, and when $c < 0$ we have sideways hyperbolas.

(b) The x-slices have (y, z) satisfying $z = y^2 - c^2$, so they are upward facing parabolas shifted down by c^2. The y-slices have (x, z) satisfying $z = -x^2 + c^2$, so they are downward facing parabolas shifted up by c^2.

(c) The graph looks like a horse saddle/pringle.

(d) First we compute $\nabla f = (-2x, 2y)$, and so we have $\nabla f(2, 0) = (-4, 0)$ and $\nabla f(0, 2) = (0, 4)$.

At the point $(2, 0)$ the fastest direction of increase is the direction of $(-4, 0)$. The fastest rate of change for directional derivatives is in the direction of the gradient, which is in the opposite direction of e_1 at $(2, 0)$. From this we conclude that the graph is going most steeply down as we travel in the x-direction (since the direction of most *decrease* will be in the opposite direction of the gradient because $D_{-v}f = -D_v f$). Furthermore, as we travel along the y-direction we see the rate of change of the z-coordinate is 0 at $(2, 0)$.

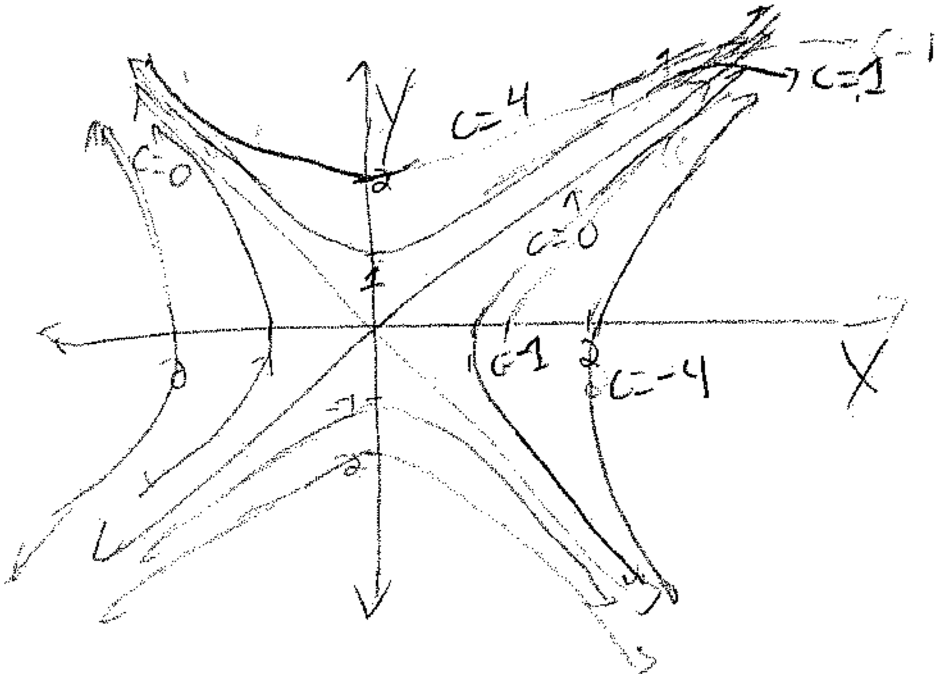
At $(0, 2)$, the graph must be going most steeply upwards in the y-direction, and the rate of change of the z-coordinate is 0 in the x-direction.

(e) The origin is a critical point because in both coordinate directions we have local extrema on the coordinates slices (even though on the graph, no local extremum occurs at the origin).

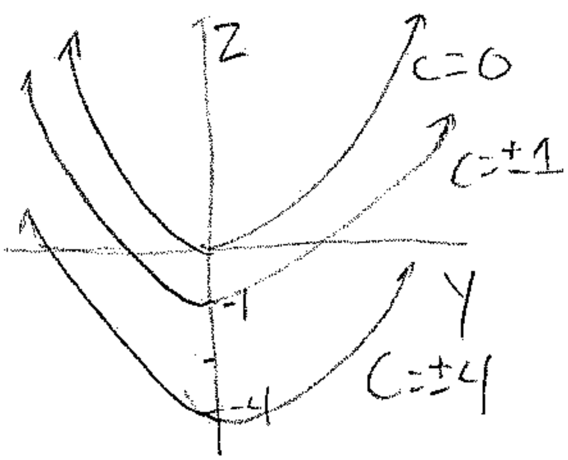
As we travel along the y-axis, we trace through an upward facing parabola on the graph with it's vertex at the origin (since $f(0, y) = y^2$). So $\frac{\partial f}{\partial y}(0, 0) = 0$ because we have a local minimum in that direction. Similarily, as we travel in the x-direction towards the origin we have a local maximum at the origin (because we now have an downward facing parabola $f(x, 0) = -x^2$). Therefore we also have $\frac{\partial f}{\partial x}(0, 0) = 0$.

More generally, we have a saddle point. Also this particular graph is referred to both as a hyperbolic paraboloid and as a *saddle*. The first term makes sense when you look at the cross sections. The second term make sense because the graph actually looks like a saddle that you would sit on.

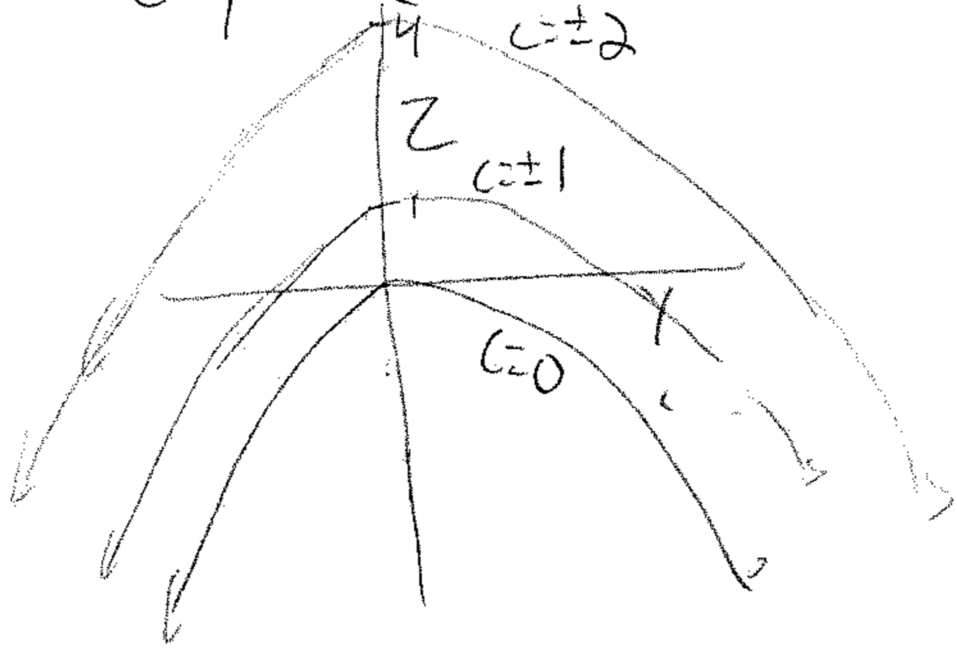
a.



b. x-slice



c. y-slice



c. Graph